

APRIL/MAY 2024

GMA12 — REAL ANALYSIS-I



Time : Three hours

Maximum : 75 marks

SECTION A — ( $10 \times 2 = 20$  marks)

Answer ALL questions.

1. Define bounded variation.
2. If  $f$  is continuous on  $[a, b]$  and if  $f'$  exists and is bounded in the interior say  $|f'(x)| \leq A$  for all  $x$  in  $(a, b)$ , prove that  $f$  is of bounded variation on  $[a, b]$ .
3. Define Riemann condition.
4. State Euler's summation formula.
5. State first mean-value theorem for Riemann-stieltjes integrals.
6. State change of variable in a Riemann integral.

7. Prove that absolute convergence of  $\sum a_n$  implies convergence.
8. State Dirichlet's test.
9. State Cauchy condition for uniform convergence of series.
10. Define mean convergence.

SECTION B — (5 × 5 = 25 marks)

Answer ALL questions.

11. (a) Let  $f$  be bounded variation on  $[a, b]$  and assume that  $C \in (a, b)$ . Show that  $f$  is of bounded variation on  $[a, c]$  and on  $[c, b]$  and we have  $V_s(a, b) = V_s(a, c) + V_s(c, b)$ .

Or

- (b) If  $f$  and  $g$  are each of bounded variation on  $[a, b]$ , prove that  $f \pm g, fg$  are bounded variation on  $[a, b]$  and we have  $V_{f \pm g} \leq V_f + V_g$  and  $V_{fg} \leq AV_f + BV_g$  where  $A = \sup \{ |g(x)| : x \in [a, b] \}, B = \sup \{ |f(x)| : x \in [a, b] \}$

(c)  $\sum_{n=1}^{\infty} f(m, n)$  and  $\sum_{n=1}^{\infty} (m, n)$  both converge absolutely.

(d) If  $A \sum_{n=1}^{\infty} f(m, n)$  and  $B_n \sum_{n=1}^{\infty} f(m, n)$  both series  $\Sigma A_n$  and  $\Sigma B_n$  converge absolutely and both have sum  $S$ . That is

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m, n) = s.$$

20. State and prove Dirichlet's test for uniform converge.

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12. (a) If  $f \in R(\alpha)$  on  $[a, b]$  show that  $\alpha \in R(f)$  on  $[a, b]$  and we have

$$\int_a^b f(x) d\alpha(x) + \int_a^b \alpha(x) df(x) = f(b)\alpha(b) - f(a)\alpha(a)$$

Or

(b) State and prove Reduction of Riemann-Stieltjes integral to a finite sum.

13. (a) State and prove second fundamental theorem of integral calculus.

Or

(b) State and prove change in a Riemann integral.

14. (a) Let  $\Sigma a_n$  be an absolutely convergent series having sum  $S$ . Prove that every rearrangement  $\Sigma a_n$  also converges absolutely and has sum  $S$ .

Or

(b) Let  $\lim_{p, q \rightarrow \infty} f(p, q) = \alpha$ , for each fixed  $p$  assume that the limit  $q \rightarrow \infty f(p, q)$  exists. Show that  $\lim_{p \rightarrow \infty} (\lim_{q \rightarrow \infty} f(p, q))$  also exists and has the value  $\alpha$ .

15. (a) Let  $\{f_n\}$  be a sequence of functions defined on a set  $S$ . Prove that there exists a function  $f$  such that  $f_n \rightarrow f$  uniformly on  $S$  if, and only if, the following condition is satisfied. For every  $\varepsilon > 0$  there exists an  $N$  such that  $M > N$  and  $n > N$  implies  $|f_n(x) - f_n(x)| < \varepsilon$  for every  $x$  in  $S$ .

Or

- (b) If  $\lim_{n \rightarrow \infty} f_n = f$  on  $[a, b]$  and if  $g \in R[a, b]$ , define  $h(x) = \int_a^x f(t) g(t) dt$ ,  $h_n(x) = \int_a^x f_n(t) g(t) dt$  if  $x \in [a, b]$ . Prove that  $h_n \rightarrow h$  uniformly on  $[a, b]$ .

SECTION C — (3 × 10 = 30 marks)

Answer any THREE questions.

16. Let  $f$  be of bounded variation on  $[a, b]$ . If  $x \in (a, b)$  Let  $V(x) = V_f(a, x)$  and Put  $V(a) = 0$ . Show that every point of continuity of  $f$  is also a point of continuity of  $V$ . The converse is also true.

17. Let  $\alpha$  on  $[a, b]$ . Show that the following three statements are equivalent

- (a)  $f \in R(\alpha)$  on  $[a, b]$   
 (b)  $f$  satisfies Riemann's condition with respect to  $\alpha$  on  $[a, b]$   
 (c)  $I(f, \alpha) = I(f, \alpha)$

18. If  $f$  is continuous on the rectangle  $[a, b] \times [c, d]$  and if  $g \in R$  on  $[a, b]$ . Show that the function  $F$

defined by equation  $F(y) = \int_a^b g(x) f(x, y) dx$ , is

continuous on  $[c, d]$ . That is if  $y_0 \in [c, d]$ , we have

$$\lim_{y \rightarrow y_0} \int_a^b g(x) f(x, y) dx = \int_a^b g(x) f(x, y_0) dx.$$

19. Let  $\Sigma f(m, n)$  be a given double series and let  $g$  be an arrangement of the double sequence  $f$  into a sequence  $G$ . Prove that

- (a)  $\Sigma G(n)$  converges absolutely if and only if,  $\Sigma f(m, n)$  converges absolutely.

(b)  $\sum_{n=1}^{\infty} G(n) = s$